Math 249 Lecture 20 Notes

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1 The Littlewood-Richardson Rule

From last time, we have the jeu de taquin coefficients c^{ν}_{λ}, where ν was a skew shape and λ was a straight shape. c^{ν}_{λ} was the number of $T \in SYT(\nu)$ such that $R(T)$ has shape λ. We have $s_\nu = \sum_{|\lambda|=|\nu|} c^{\nu}_{\lambda} s_\lambda$, which gave us

$$c^{\mu/\lambda}_{\kappa} = \left\langle s_{\mu/\lambda}, s_\kappa \right\rangle = \left\langle s_\mu, s_\kappa s_\lambda \right\rangle = s^{\mu}_{\kappa,\lambda},$$

the Littlewood-Richardson coefficients.

1.1 Yamanouchi tableau

This are also called "lattice" or "ballot" tableau [bad notation]

Definition 1.1. The *reading word* of a skew tableau is a word containing the entries of the Young tableau in reading order (left to right on each row, reading rows from top to bottom).

Example 1.1. The tableau

$$\begin{array}{|c|cc} \hline 7 & \multicolumn{1}{c}{} & \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c|}{5} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{3} \\ \cline{3-3} \end{array}$$

has reading word 7 2 5 3.

Definition 1.2. A SSYT T is *Yamanouchi*[1] if every tail of its reading word has weights that form a valid partition. That is, for each tail of the reading word, for each letter i, the number of letters of i in the tail should be at least the number of letters $i+1$ in the tail.

These are also sometimes called

[1] These are also sometimes called "lattice" or "ballot" tableau. Lattice has too many meanings in math already, and Professor Haiman prefers to to call them Yamanouchi rather than "ballot" tableau.

Example 1.2. Consider the skew tableau

$$\begin{array}{|c|c|c|} \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline & & 1 \\ \hline \end{array}$$

The reading word of this tableau is 231121. The tails of this are 1, 21, 121, etc. 1 forms a partition. 21 forms a partition because there are not more 2s than 1s. 121 forms a parition for the same reason. If we continue checking, we see that this tableau is Yamanouchi.

On the other hand, the following tableau is not Yamanouchi:

$$\begin{array}{|c|c|c|} \hline 2 & 2 & \\ \hline 1 & 1 & 3 \\ \hline & & 1 \\ \hline \end{array}$$

It has reading word 2 2 1 1 3 1, and the tail 3 1 is not a valid partition because there is a 3 but no 2.

Proposition 1.1. *There is exactly one Yamanouchi tableau of a given straight shape λ.*

Proof. If λ is a straight shape, we must have 1 in the bottom right corner; then we must have 1s in every place in the bottom row. We then have the same for the next row up, but instead we must have 2 in every place in the 2nd row (since SSYTs are strictly increasing in columns). Continuing this reasoning up the rows, we see that each straight shape has a unique Yamanouchi tableau. □

We call the Yamanouchi tableau with shape ν and weight λ $\mathrm{Yam}(\nu, \lambda)$. So if ν is a partition and $\nu \neq \lambda$, there is no Yamanouchi tableau. If ν is a partition and $\nu = \lambda$, then we have a tableau like this:

$$\begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 3 & 3 & & \\ \hline 2 & 2 & & \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Lemma 1.1. *Jeu de taquin preserves Yamanouchi tableau.*

Proof. We just have to check that this is okay when we move a hole. We treat the case of forward slides; a similar argument works for reverse slides. There are two cases: the hole moves up, or the hole moves right. If the hole moves to the right, the reading word does not change. When the hole moves up, we only need to look at the two rows that changed.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline w_1 & \cdots & w_n & i & x_1 & \cdots & x_m & & \\ \hline y_1 & \cdots & y_n & \blacksquare & z_1 & \cdots & \cdots & \cdots & z_\ell \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline w_1 & \cdots & w_n & \blacksquare & x_1 & \cdots & x_m & & \\ \hline y_1 & \cdots & y_n & i & z_1 & \cdots & \cdots & \cdots & z_\ell \\ \hline \end{array}$$

The two relevant reading words look like

$$\text{Original: } w_1 \cdots w_n \, i \, x_1 \cdots x_m \, y_1 \cdots y_n \, z_1 \cdots z_\ell \cdots = W \, i \, X \, Y \, Z \cdots$$

$$\text{Altered: } w_1 \cdots w_n \, x_1 \cdots x_m \, y_1 \cdots y_n \, i \, z_1 \cdots z_\ell \cdots = W \, X \, Y \, i \, Z \cdots .$$

Since we are assuming that the original tableau is Yamanouchi, the only condition we could violate is the condition that the number of i letters should not be bigger than the number of $(i-1)$ letters in some tail. This is semi-standard, so we have that letter by letter, $X > Z \geq i$. Then x has no is. Similarly, $Y < W$ letter by letter, and $Y < W \leq i$, so Y has no is.

In Z, there are at least 1 more $i-1$ than i letters (because the Yamanouchi condition is satisfied with the tail $i \, X \, Y \, Z \cdots$), so if we move an i down, then we dont violate the condition for any tails ending within Z or the tail $i \, Z \cdots$. Since there are no is in X or Y, any tails ending within them still satisfy the Yamanouchi condition. These are the only places where tails of the reading words may differ, so we are done. □

1.2 The Littlewood-Richardson rule

We get a nice corollary from the above lemma.

Corollary 1.1 (Littlewood-Richardson rule). *Let λ be a partition, and let ν be a skew shape. Then*

$$c_\lambda^\nu = |\mathrm{Yam}(\nu, \lambda)| \,,$$

the number of Yamanouchi tableau of shape ν and weight λ.

Example 1.3. The Pieri rules are a specific case of the Littlewood-Richardson rule:

$$h_k s_\lambda = \sum_\mu c^\mu_{(k),\lambda} s_\mu = \sum_\mu c^{\mu/\lambda}_{(k)} s_\mu,$$

where

$$c^{\mu/\lambda}_{(k)} = |\mathrm{Yam}(\mu/\lambda, (k))| = |\{\mu/\lambda : \mu/\lambda \text{ is a horizontal strip}\}| \,.$$

$$e_k s_\lambda = \sum_\mu c^\mu_{(1^k),\lambda} s_\mu = \sum_\mu c^{\mu/\lambda}_{(1^k)} s_\mu,$$

where

$$c^{\mu/\lambda}_{(1^k)} = |\,\mathrm{Yam}(\mu/\lambda, (1^k))| = |\{\mu/\lambda : \mu/\lambda \text{ is a vertical strip}\}| \,.$$

For this case, the Yamanouchi condition means that the reading word is $k, k-1, \ldots, 1$, which makes μ/λ a vertical strip.

Example 1.4. We can try calculating $s_{2,1}^2$ by counting how many Yamanouchi tableau there are of a given shape.

$$s_{2,1}^2 = \sum_\mu c^{\mu/(2,1)}_{(2,1)} s_\mu = s_{4,2} + s_{4,1,1} + s_{3,3} + 2 s_{3,2,1} + s_{2,2,2} + s_{3,1,1,1} + s_{2,2,1,1}.$$